Leakage Rate as a Measure of Continuous-Time Stochastic Set Invariance

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Abstract— **Probabilistic set invariance is a concept that measures at which probability the given set is invariant under a stochastic process. While this concept successfully measures the level of invariance of a given set, its major limitations include excessive conservativeness and lack of consideration of the prior distribution of the state. In this paper, we introduce a modified notion of probabilistic set invariance that takes into account the starting distribution of the stochastic process. The probability of never escaping the given set, i.e., the survival rate, is defined as a function of the initial distribution and the length of the time window, and its infinitesimal-time behavior is analyzed. For Itô diffusion processes, we find that the decaying rate of survival rate, which we call the *leakage rate*, can be analytically evaluated through a surface integral formula on the boundary of the set given the probability distribution of state and the update rule. We validate the formula through a numerical example, in which the simulation result well matches the analytical prediction.**

I. Introduction

Set invariance is an important topic in control systems engineering that directly relates to ensuring safety of a system [1]. Including the classic Lyapunov methods [2], [3], invariance-based frameworks such as control barrier functions [4], [5] and Hamilton-Jacobi reachability analysis [6] have proven to work well in deterministic settings where the nominal model and the upper bound of the modeling error are all known. Such methods are recently being applied to more complex tasks, for example, collision avoidance for robots [7], [8].

However, many real-world systems are subject to disturbance whose magnitude is unbounded, for example, Gaussian noise models. In the typical case where actuation limits are present, completely (with probability 1) preventing violation of the safety constraints is in general not possible, and invariance can only be guaranteed in a probabilistic manner. The concept of *probabilistic invariant sets* was introduced for this purpose: A subset C of the state space is said to be ϵ-probabilistic invariant ($\epsilon \in [0, 1)$ is the *escape probability*) if the probability of a trajectory residing in C during the given time window T is at least $1 - \epsilon$ for *any* deterministic initial state in C, where T may be given infinite [9].

Various prior works studied probabilistic invariance. The works [9]–[12] proposed ways to evaluate ϵ given C or vice versa for discrete-time systems. For the continuous-time case, especially Itô diffusion processes, barrier-function-based approaches are recently gaining interest to construct probabilistic invariant sets [13]–[17].

The most significant drawback of the existing probabilistic invariance framework, however, is that it tends to be very conservative, because only the worst among all escaping probabilities from C is used as the invariance measure. The chance of escaping C naturally appears to be the largest for initial conditions on the set boundary, but the actual probability of the starting state being on the boundary, which is in most cases infinitesimal, is not taken into account. Moreover, the time horizon T should be specified before execution, which is another limitation.

In this paper, as a first step towards overcoming these shortcomings and building an initial-distribution-aware feedback strategy preserving probabilistic invariance for continuous-time stochastic systems, we instead evaluate the escape probability over the given *distribution* of starting state, not for every deterministic initial condition in C. We consider chance of survival as a function of time, i.e., the probability of the trajectory never escaping C before the given timepoint. While it is obvious that this survival rate gradually decays as time flows, it is very natural to claim that its *rate of decay* measures the level of invariance of C at a specific time. The notion of *leakage rate* from the set C is introduced, which directly relates to the decaying speed of the survival rate and measures how invariant the set C is. We derive a formula to calculate the leakage rate Itô diffusion processes, and the formula is validated using a numerical simulation.

The advantages of the proposed framework compared to probabilistic invariant sets are: it takes into account the actual starting distribution and therefore is not conservative when estimating the survival rate; and there is no need to specify prediction length T, as we consider infinitesimal time window. We expect that the concept of leakage rate can be useful in practical applications where accurate evaluation of the safety risk is important, for example, aerial vehicle motion planning and control [18], [19] or autonomous driving [20]–[22].

Notation

For brevity of the derivation, we define $\phi : \mathbb{R} \cup \{\pm\infty\} \to \mathbb{R}_{\geq 0}$ and $\Phi : \mathbb{R} \cup \{\pm\infty\} \to [0, 1]$ as the probability density and the cumulative distribution functions of the unit normal distribution:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},\ \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \tag{1}$$

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for finite x, and the function values are defined commonsensically for infinite x. To handle convergence of the limits, we use the big-O and small-o notations. For real-valued functions $f_L(t)$, $f_R(t)$ and $g(t)$,

$$f_L(t) = f_R(t) + O(g(t)) \tag{2}$$

if $|f_L(t) - f_R(t)| \leq c|g(t)|,\ \forall t \in (0, T]$ for some positive constants c and T. We write

$$f_L(t) = f_R(t) + o(g(t)) \tag{3}$$

if the same holds for *any* positive constant c.

II. Leakage Rate: Definition

In this paper, we consider a continuous-time stochastic process $\{X_t\}_{t\in\mathbb{R}_{\geq 0}}$ on state space $\mathbb{X}$ indexed by time $t \in \mathbb{R}_{\geq 0}$ with Markov property. Let C be a measurable subset of $\mathbb{X}$ whose leakage rate we want to measure, i.e., how invariant the set C is. For a given timepoint t, a trajectory is considered lost invariance if it has at least once visited outside of C within $[0, t]$. To that end, we make use of the concept of *killed process*.

Definition 1 (Killed Process)**.** Consider a stochastic process Y_t on $\mathbb{X} \cup \{K\}$ defined for $t \in \mathbb{R}_{\geq 0}$, where $K \notin \mathbb{X}$ is the *coffin state*. Let $E(Y_t, t)$ be the (possibly random) event on which the process is *killed*. The process Y_t is a killed process if $Y_\tau = K$ for all $\tau \geq t$ given $E(Y_t, t)$ for some $t \in \mathbb{R}_{\geq 0}$.

Define Y_t as the following stochastic process on $\mathbb{X} \cup \{K\}$:

$$Y_t = \begin{cases} X_t & \text{if } X_\tau \in C\ \forall \tau \in [0, t], \\ K & \text{else.} \end{cases} \tag{4}$$

Then, Y_t is a killed process with the Markov property. Whenever X_t escapes the set C, Y_t is immediately removed from $\mathbb{X}$ and permanently stored at the coffin state K.

The probability distribution of Y_t can be described using two functions, $p : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\Gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$:

$$\mathbb{P}[Y_t \in S] = \int_S p(x, t)dx \tag{5}$$

for any measurable subset S of C, and

$$\Gamma(t) = \mathbb{P}[Y_t \neq K] = \int_C p(x, t)dx, \tag{6}$$

where the integral can be the summation for discrete spaces. We call $\Gamma(t)$ the *survival rate* hereafter. We denote the conditional distribution of Y_t given $Y_t \neq K$ using $q(x, t)$. It can be easily found that $q(x, t) = p(x, t)/\Gamma(t)$.

The probability of Y_t being alive, i.e., the survival rate $\Gamma(t) := \mathbb{P}[Y_t \neq K]$, measures how invariant the set C is and plays a similar role with the measure $1 - \epsilon$ for probabilistic invariant sets. Indeed, the behavior of $\Gamma(t)$ depends on the initial distribution X_0, and it decays towards zero as $t \to \infty$. In this paper, we are mainly interested in how $\Gamma(t)$ behaves as time flows, especially its time derivative $\dot{\Gamma}(t)$.

Definition 2 (Leakage Rate)**.** The leakage rate of X_t from set C is defined as the limit

$$\gamma(t) := \lim_{s \searrow 0} \frac{\mathbb{P}\left[Y_{t+s} = K | Y_t \neq K\right]}{s}. \tag{7}$$

The numerator is the conditional probability of X_t escaping the set C at least once during the interval $[t, t+s]$, given X_t has never escaped C before.

It should be noted that the limit might diverge depending on the distribution Y_t. Later in this paper, we will also deduce a sufficient condition for its convergence. Under the assumption that $\gamma(t)$ has a finite value, it is straightforward to find that $\gamma(t) \geq 0$, and $\Gamma(t)$ has a right derivative

$$\dot{\Gamma}(t) = -\Gamma(t)\gamma(t). \tag{8}$$

III. Computation of Leakage Rate

A. Motivating Example: Discrete State Space

Before examining the continuous state space case (which is the main contribution of this paper), as a motivating example, we first look into the case where the state space has N $(< \infty)$ elements. Here, the state space takes the form $\mathbb{X} = \{x_i : i \in \{1, \cdots, N\}\}$, where $x_i = x_j$ if and only if $i = j$. The distribution X_t can be written using functions $p_i(t) = \mathbb{P}[X_t = x_i]$, and the *transition rate* from x_i to x_j $(i \neq j)$ is defined as

$$Q_{ij} = \lim_{s \searrow 0} \frac{\mathbb{P}\left[X_{t+s} = x_j \middle| X_t = x_i\right]}{s}. \tag{9}$$

Then, the dynamics of the distribution over X is written as follows:

$$\dot{p}_i(t) = -\sum_{j \neq i} Q_{ij} p_i(t) + \sum_{j \neq i} Q_{ji} p_j(t). \tag{10}$$

Let $C = \{x_i : i \in I_C\}$ where I_C is a subset of $\{1, \cdots, N\}$. It can be found that

$$\gamma(t) = \frac{\sum_{i \in I_C, j \notin I_C} Q_{ij} p_i(t)}{\sum_{i \in I_C} p_i(t)} \tag{11}$$

if $\Gamma(t) > 0$, $\gamma(t) = 0$ if $\Gamma(t) = 0$. Note that when evaluating $\gamma(t)$, the probability of "temporarily visiting outside and then returning back" within the infinitesimal interval should also be taken into account. However, we omit the detailed explanation here, since it is not directly related to the main content of this paper.

B. Computation of Leakage Rate for Itô Diffusion Processes

We will now consider the same problem on an Itô diffusion process on a continuous Euclidean state space $\mathbb{X} = \mathbb{R}^n$. We point out that, although we simply assumed Euclidean state space here for notational brevity, the concept can be easily applied *mutatis mutandis* to finite-dimensional Riemannian manifolds.

Firstly, we assume the set of interest C is a closed subset of $\mathbb{R}^n$ that can be described by

$$C = \{x : h(x) \geq 0\}, \tag{12}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^2$ function that satisfies the following regularity conditions:

$$\begin{aligned} \text{Int } C &= \{x : h(x) > 0\} \\ \partial C &= \{x : h(x) = 0\} \\ \partial_x h(x) &\neq 0, \ \forall x \in \partial C, \end{aligned} \tag{13}$$

where Int C and ∂C are the interior and boundary of C, respectively. We assume X_t is an Itô diffusion process which satisfies the following stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{14}$$

where W_t is the n_w-dimensional Wiener process. The drift and diffusion terms $\mu : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n_w}$ are assumed to be Lipschitz continuous functions. Define Y_t as (4) so that whenever X_t reaches the boundary of C for the first time, Y_t is removed from C and permanently stored at the coffin state K.

Theorem 1 (Leakage Rate for Itô Diffusion Processes)**.** *The leakage rate $\gamma(t)$ has a finite value if there exists a continuous function $m_t : C \to \mathbb{R}_{\geq 0}$ such that $q(x,t) = m_t(x)h(x)$ everywhere on C, and*

$$\gamma(t) = \int_{\partial C} \frac{m_t(x) \left\| \partial_x h(x) \cdot \sigma(x) \right\|^2}{2 \left\| \partial_x h(x) \right\|} dS, \tag{15}$$

where $\int_{\partial C}(\cdots)dS$ is the surface integral over the boundary of C.

Proof. Since the dynamics (14) is time invariant, without loss of generality, we can let $t = 0$ and $\Gamma(0) = 1$, so that $p(x,0) = q(x,0)$ describes the initial probability distribution X_0. Then, the leakage rate at $t = 0$ can be written as

$$\gamma(0) = \lim_{t \searrow 0} \frac{\mathbb{P}\left[\min_{\tau \in [0,t]} h(X_\tau) < 0\right]}{t}. \tag{16}$$

Below, we use $m(x)$ and $p(x)$ as the shorthand notations for $m_0(x)$ and $q(x,0) = p(x,0)$. The probabilities $\mathbb{P}[\cdot]$ with no condition specified are measured with respect to the initial distribution X_0.

We start from the following lemma. See the appendix for its proof.

Lemma 1. *Let X_t be the stochastic process driven by (14). For any deterministic starting state $X_0 = x$ such that $h(x) \geq 0$,*

$$\mathbb{P}\left[\min_{\tau \in [0,t]} h(X_\tau) < 0 \middle| X_0 = x\right] = 2\Phi(b(t,x)) + o(t) \tag{17}$$

where $b(t,x) = -\frac{h(x)}{\left\|\partial_x h(x) \cdot \sigma(x)\right\| \sqrt{t}} \in \{-\infty\} \cup \mathbb{R}_{\leq 0}$.

The probability of reaching the set boundary ∂C within time interval $[0,t]$ can be evaluated using the following integral.

$$\begin{aligned} &\mathbb{P}\left[\min_{\tau \in [0,t]} h(X_\tau) < 0\right] \\ &= \int_C p(x) \cdot \mathbb{P}\left[\min_{\tau \in [0,t]} h(X_\tau) < 0 \middle| X_0 = x\right] dx \\ &= \int_C p(x) \cdot 2\Phi(b(t,x))dx + o(t) \end{aligned} \tag{18}$$

Let $C_{[a,b]} = \{x : a \leq h(x) \leq b\}$, $C_{(a,b)} = \{x : a < h(x) < b\}$ and $\partial C_a = \{x : h(x) = a\}$ for $a, b \geq 0$. According to the regularity condition $\partial_x h(X) \neq 0 \ \forall x \in \partial C$ and the assumption that h is $\mathcal{C}^2$, there exists a positive real number ϵ such that

$$\partial_x h(x) \neq 0, \quad \forall x \in C_{[0,\epsilon]}. \tag{19}$$

Let $\delta = \min\{\epsilon, t^{1/4}\}$ and split the integral in (18) into two parts as follows:

$$\int_C p(x) \cdot 2\Phi(b(t,x))dx = \underbrace{\int_{C_{[0,\delta]}} (\cdot)dx}_{(A)} + \underbrace{\int_{C_{(\delta,\infty)}} (\cdot)dx}_{(B)}. \tag{20}$$

To evaluate (A), we use the fact that

$$\int_{C_{[0,y]}} \beta(x, h(x))dx = \int_0^y \int_{\partial C_z} \frac{\beta(x,z)}{\left\|\partial_x h(x)\right\|} dSdz \tag{21}$$

for any continuous function $\beta : C \times \mathbb{R} \to \mathbb{R}$ if $\partial_x h(x) \neq 0$ for all $x \in C_{[0,y]}$, which holds for $y = \delta$. Let

$$\beta(x,z) = m(x)z \cdot 2\Phi\left(-\frac{z}{\left\|\partial_x h(x) \cdot \sigma(x)\right\| \sqrt{t}}\right) \tag{22}$$

and apply the coordinate transform $\zeta\sqrt{t} = z$. Then, (A) becomes

$$\begin{aligned} &\frac{(A)}{t} \\ &= \int_0^{\delta'} \int_{\partial C_{\zeta\sqrt{t}}} \frac{2m(x)}{\left\|\partial_x h(x)\right\|} \zeta \Phi\left(-\frac{\zeta}{\left\|\partial_x h(x) \cdot \sigma(x)\right\|}\right) dSd\zeta, \end{aligned} \tag{23}$$

where $\delta' = \delta/\sqrt{t}$. Notice that the integral

$$\int_{\partial C_b} \frac{2m(x)}{\left\|\partial_x h(x)\right\|} \zeta \Phi\left(-\frac{\zeta}{\left\|\partial_x h(x) \cdot \sigma(x)\right\|}\right) dS \tag{24}$$

as a function of $b \in \mathbb{R}_{\geq 0}$ is right continuous at $b = 0$. Taking the limit $t \searrow 0$, we have $\delta' \to \infty$ and $\zeta\sqrt{t} \to 0$, and thus the integral becomes

$$\begin{aligned}
&\lim_{t\searrow 0}\frac{(A)}{t}\\
&=\int_0^\infty\int_{\partial C}\frac{2m(x)}{\|\partial_x h(x)\|}\zeta\Phi\left(-\frac{\zeta}{\|\partial_x h(x)\cdot\sigma(x)\|}\right)dSd\zeta\\
&=\int_{\partial C}\int_0^\infty\frac{2m(x)}{\|\partial_x h(x)\|}\zeta\Phi\left(-\frac{\zeta}{\|\partial_x h(x)\cdot\sigma(x)\|}\right)d\zeta dS\\
&=\int_{\partial C}\frac{2m(x)\|\partial_x h(x)\cdot\sigma(x)\|^2}{\|\partial_x h(x)\|}\int_0^\infty\xi\Phi(-\xi)d\xi dS\\
&=\int_{\partial C}\frac{m(x)\|\partial_x h(x)\cdot\sigma(x)\|^2}{2\|\partial_x h(x)\|}dS.
\end{aligned}\tag{25}$$

To obtain the first equality, the continuity property of (24) was used, after which the two variables x and ζ are decoupled and the order of integration can be swapped without further modification (the second equality). The coordinate transform $\xi\cdot\|\partial_x h(x)\cdot\sigma(x)\|=\zeta$ was used to arrive at the third equality, and finally the (inner) improper integral $\int_0^\infty\xi\Phi(-\xi)d\xi=1/4$ was evaluated.

For all $x\in C_{(\delta,\infty)}$, $h(x)>t^{1/4}$. Since Φ is a strictly increasing function, for all $x\in C_{(\delta,\infty)}$,

$$\begin{aligned}
\Phi(b(t,x))&<\Phi\left(-\frac{t^{1/4}}{\|\partial_x h(x)\cdot\sigma(x)\|\sqrt{t}}\right)\\
&=\Phi\left(-\frac{1}{\|\partial_x h(x)\cdot\sigma(x)\|t^{1/4}}\right)
\end{aligned}\tag{26}$$

and therefore

$$\begin{aligned}
\lim_{t\searrow 0}\frac{\Phi(b(t,x))}{t}&=\lim_{v\to\infty}v^4\Phi(b(v^{-4},x))\\
&\le\lim_{v\to\infty}v^4\Phi\left(-\frac{v}{\|\partial_x h(x)\cdot\sigma(x)\|}\right)\\
&=0,
\end{aligned}\tag{27}$$

where we substituted $v^{-4}=t$ here. Therefore, we have

$$\lim_{t\searrow 0}\frac{(B)}{t}=0.\tag{28}$$

Back to the definition of γ, observe

$$\begin{aligned}
\gamma(0)&=\lim_{t\searrow 0}\frac{1}{t}\int_C p(x)\cdot\mathbb{P}\left[h(X_t)<0\middle|X_0=x\right]dx\\
&=\lim_{t\searrow 0}\frac{(A)+(B)+o(t)}{t}=\lim_{t\searrow 0}\frac{(A)}{t}+\lim_{t\searrow 0}\frac{(B)}{t}
\end{aligned}\tag{29}$$

and the proof is complete. □

If $p(x,t)\neq 0$ on ∂C, $\gamma(t)$ does not have a finite value and therefore $\Gamma(t)$ experiences a sudden drop in its value. In this case,

$$\eta(t):=\lim_{s\searrow 0}\frac{\mathbb{P}\left[Y_{t+s}=K|Y_t\neq K\right]}{\sqrt{s}}\tag{30}$$

can alternatively measure how fast $\Gamma(t)$ drops. Following the similar procedure as Theorem 1, we obtain the following corollary.

Corollary 1. *Suppose there exists a positive constant ϵ such that $p(x,t)$ is continuous on $C_{[0,\epsilon]}$. Then,*

$$\eta(t)=\int_{\partial C}\sqrt{\frac{2}{\pi}}\frac{p(x,t)\|\partial_x h(x)\cdot\sigma(x)\|^2}{\|\partial_x h(x)\|}dS.\tag{31}$$

Proof. In the proof of Theorem 1, instead of (22), let

$$\beta(x,z)=p(x)\Phi\left(-\frac{z}{\|\partial_x h(x)\cdot\sigma(x)\|\sqrt{t}}\right).\tag{32}$$

□

C. Implications and Remarks

Intuitively, $\gamma(t)$ should be invariant under the choice of h to represent the set C. For example, let $h_1(x)=n(x)h(x)$ for some function $n:C\to\mathbb{R}$ and assume that there exists $\epsilon>0$ such that $n(x)\ge\epsilon$ for all $x\in C$. The new h_1 gives the same C as before and satisfies the regularity conditions abovementioned. Now, observe that

$$\begin{aligned}
\partial_x h_1(x)&=n(x)\partial_x h(x)+h(x)\partial_x n(x)\\
&=n(x)\partial_x h(x),\quad\forall x\in\partial C
\end{aligned}\tag{33}$$

since $h(x)=0$ everywhere on ∂C, and let $q(x,t)=m_{1,t}(x)h_1(x)$ so that $m_{1,t}(x)=m_t(x)/n(x)$. Then, the new leakage rate $\gamma_1(t)$ obtained using h_1 evaluates to

$$\begin{aligned}
\gamma_1(t)&=\int_{\partial C}\frac{m_{1,t}(x)\|\partial_x h_1(x)\cdot\sigma(x)\|^2}{2\|\partial_x h_1(x)\|}dS\\
&=\int_{\partial C}\frac{\frac{m_t(x)}{n(x)}\cdot n(x)^2\|\partial_x h(x)\cdot\sigma(x)\|^2}{2n(x)\cdot\|\partial_x h(x)\|}dS\\
&=\gamma(t),
\end{aligned}\tag{34}$$

which matches our intuition.

It is interesting that $\gamma(t)$ does not explicitly depend on the drift term μ. This is because within an infinitesimal time window, the influence of the diffusion term *dominates* that of the drift term. Informally speaking, among the two terms contributing to the state displacement

$$X_t-X_0=\int_0^t\mu(X_\tau)d\tau+\int_0^t\sigma(X_\tau)dW_\tau,\tag{35}$$

the magnitude of the drift-related term depends linearly with respect to the size of the time window (i.e., $\mathbb{E}\left\|\int_0^t\mu(X_\tau)d\tau\right\|=O(t)$), while the diffusion-related term is proportional to the square root (i.e., $\mathbb{E}\left\|\int_0^t\sigma(X_\tau)dW_\tau\right\|=O(\sqrt{t})$), as $t\searrow 0$.

This implies that, in feedback synthesis problems where μ depends on the user-given input but σ cannot be directly modulated, probabilistic safety guarantee should be achieved by regulating $m_t(x)$ on the set boundary, where $m_t(x)$ indicates the *steepness* of q is on ∂C. As seen in the integrand $m_t(x)\|\partial_x h(x)\cdot\sigma(x)\|^2/2\|\partial_x h(x)\|$, $m_t(x)$ should be kept small where $\sigma(x)$ is larger (and $m_t(x)$ can be bigger where $\sigma(x)$ is small) for the same level of safety. Smaller $m_t(x)$ values imply that the drift term μ pushes the system *more aggressively* into C, and this well matches the common sense that the control should be more aggressive (smaller m_t) under larger uncertainty (big σ).

It is very natural to expect that the m_t can be regulated using the recent barrier-function-based approaches. Aligned with the recent work on stochastic extensions of control barrier functions [4], [13]–[15], we conjecture that a function $\alpha : \mathbb{R} \times \mathbb{R}^{n\times n_w} \to \mathbb{R}$ can be found so that

$$\mathcal{A}h(x) + \alpha(h(x), \sigma(x)) \geq 0 \tag{36}$$

ensures $m(x) \leq M$, $\forall x \in \partial C$ where $M \in \mathbb{R}_{\geq 0}$ is a preset upper bound on $m(x)$.[1] Checking whether this holds is left as a future work.

IV. Long-Term Behavior

In this section, we examine the long-term (time window being not infinitesimal) behavior of Y_t conditioned on $Y_t \neq K$, i.e., $q(x,t)$. We also discuss the asymptotic behavior of $q(x,t)$.

Since Y_t does not jump to or from the coffin state K given $Y_t \in \operatorname{Int} C$, it follows the same SDE (14) as X_t within $\operatorname{Int} C$, and $p(x,t)$ therefore satisfies the Fokker-Planck equation:

$$\begin{aligned}&\partial_t p(x,t)\\ &= \underbrace{-\sum_{i=1}^{n}\partial_i(\mu_i(x)p(x,t)) + \sum_{i=1}^{n}\sum_{j=1}^{n}\partial_i\partial_j(D_{ij}(x)p(x,t))}_{\text{(RHS)}}.\end{aligned} \tag{37}$$

where the operator ∂_i denotes the partial derivative with respect to the i-th component of x, μ_i is the i-th component of $\mu(x)$, and $D_{ij}(x)$ is the ij-th element of the $n\times n$ matrix $\frac{1}{2}\sigma(x)\sigma(x)^\top$. Suppose that $\gamma(t)$ is continuous and has a finite value for every $t \in \mathbb{R}_{\geq 0}$. Since $q(x,t) = p(x,t)/\Gamma(t)$ and $\dot{\Gamma}(t) = -\gamma(t)\Gamma(t)$, we have

$$\begin{aligned}\partial_t q(x,t) = \partial_t\left(\frac{p(x,t)}{\Gamma(t)}\right) &= \frac{\partial_t p(x,t)}{\Gamma(t)} - \frac{\dot{\Gamma}(t)p(x,t)}{\Gamma(t)^2}\\ &= \frac{\text{(RHS)}}{\Gamma(t)} + \gamma(t)q(x,t)\end{aligned} \tag{38}$$

and thus

$$\begin{aligned}&\partial_t q(x,t) - \gamma(t)q(x,t)\\ &= -\sum_{i=1}^{n}\partial_i(\mu_i(x)q(x,t)) + \sum_{i=1}^{n}\sum_{j=1}^{n}\partial_i\partial_j(D_{ij}(x)q(x,t)),\end{aligned} \tag{39}$$

which is the Fokker-Planck equation with a decay-compensation term $-\gamma(t)q(x,t)$.

Further, we examine the case where the distribution of Y_t conditioned by $Y_t \neq K$ is kept time invariant, i.e., $\partial_t q(x,t) = 0$. We denote this steady-state solution by $q(x)$.

[1] Here,

$$\mathcal{A}h(x) \coloneqq \lim_{t\searrow 0}\frac{\mathbb{E}[h(X_t)|X_0 = x] - h(x)}{t}$$

is the infinitesimal generator of h.

It is obvious that $\gamma(t)$ is also kept constant in this steady-state case, so let $\gamma(t) = \gamma$. Then, $q(x)$ should satisfy

$$\begin{aligned}&-\gamma q(x)\\ &= -\sum_{i=1}^{n}\partial_i(\mu_i(x)q(x)) + \sum_{i=1}^{n}\sum_{j=1}^{n}\partial_i\partial_j(D_{ij}(x)q(x)),\end{aligned} \tag{40}$$

and since γ is finite, $q(x)$ is subject to the boundary condition

$$q(x) = 0, \quad \forall x \in \partial C. \tag{41}$$

Note that γ depends on the density $q(x)$, and thus this boundary value problem is not linear. The following are open topics requiring further research:

- existence and uniqueness of solution to the time-varying and steady-state partial differential equations (PDEs),
- convergence of $q(x,t)$ to $q(x)$,
- development of an algorithm that efficiently solves the PDEs.

V. Numerical Example

In this section, we look into a simple numerical example to validate the main result. Let X_t be a stochastic process in $\mathbb{R}$ driven by

$$dX_t = -X_t dt + dW_t. \tag{42}$$

Let $C = \{x \in \mathbb{R} : h(x) = 1 - x^2 \geq 0\}$. The steady-state Fokker-Planck equation (40) then yields the following boundary value problem form:

$$\begin{aligned}(1+\gamma)q(x) + xq'(x) + \frac{1}{2}q''(x) = 0\\ q(-1) = q(1) = 0,\end{aligned} \tag{43}$$

where $(\cdot)'$ denotes the derivative with respect to x. To evaluate γ, let $m(x) = \lim_{s\to x} q(x)/h(x)$ to get $m(-1) = q'(-1)/2$ and $m(1) = -q'(1)/2$. With that, γ evaluates to

$$\gamma = \frac{m(-1)\cdot 2^2}{2\cdot 2} + \frac{m(1)\cdot 2^2}{2\cdot 2} = \frac{q'(-1) - q'(1)}{2}. \tag{44}$$

Since the state space is one-dimensional and the surface integral turns into the sum of the integrand values at the two boundary points of C.

Remark 1. In this special case, there is another way to obtain (44) using the property $\int_{-1}^{1} q(x)dx = 1$ and the boundary condition $q(\pm 1) = 0$:

$$\begin{aligned}1 &= \int_{-1}^{1} q(x)dx\\ &= 1\cdot q(1) - (-1)\cdot q(-1) - \int_{-1}^{1} xq'(x)dx\\ &= \int_{-1}^{1}\left((1+\gamma)q(x) + \frac{1}{2}q''(x)\right)dx\\ &= 1 + \gamma + \frac{q'(1) - q'(-1)}{2}.\end{aligned} \tag{45}$$

Using the shooting method and MATLAB's `ode45` solver, we found a numerical solution with $\gamma = 1.596$ that satisfies the differential equation and the boundary conditions, as shown in Fig. 1. Initial conditions are sampled from this

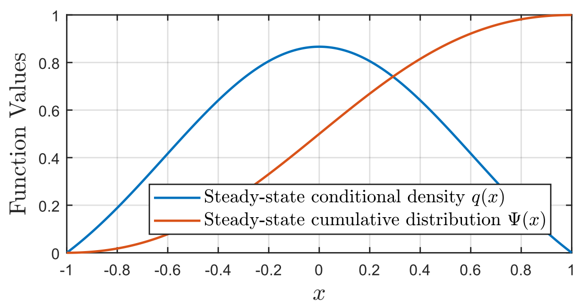

Fig. 1. The plot of the steady-state survival-conditioned distribution $q(x)$ and its cumulative counterpart $\Psi(x) = \int_{-1}^{x} q(u)du$, obtained by solving the boundary value problem (43) using the shooting method.

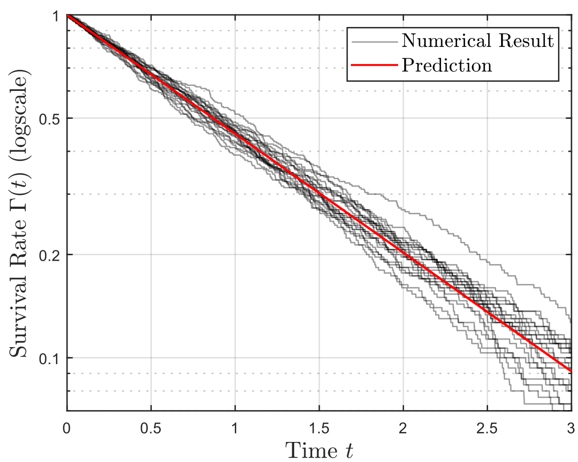

Fig. 2. The survival rate $\Gamma(t)$. Each gray line depicts the number of surviving trials divided by the number of total trials for a batch, the red line is the predicted survival rate $\Gamma(t) = e^{-\gamma t}$. The numerical result well matches the analytical prediction.

distribution.[2] Simulation was done on the interval $t \in [0, 3]$ using the Euler-Maruyama method with step size $\Delta t = 10^{-4}$. The simulation comprises 20 batches in total, where each batch consists of 300 trials. Fig. 2 shows the *survival rate* $\Gamma(t)$. As seen in the plot, the prediction $\Gamma(t) = e^{-\gamma t}$ well matches the measured. Fig. 3 showcases that, as predicted using (43), the distribution of the surviving states stays stationary, although the uncertainty in measurement escalates as time flows due to the decrement in the number of surviving samples.

VI. Conclusion

In this work, we defined the concept of leakage rate and proposed a surface integral that evaluates the value of leakage rate for Itô diffusion processes. The surface integral depends on the gradient of the probability density function of the survival-conditioned distribution of state, and the magnitude of the diffusion term on the boundary of the set. A numerical simulation for a simple SDE-driven process was conducted

[2] Write the cumulative distribution function as $\Psi(x) = \int_{-1}^{x} q(u)du$ and let U be a uniformly-distributed random variable on the interval $[0, 1]$. The random variable $\Psi^{-1}(U)$ follows the distribution X_0.

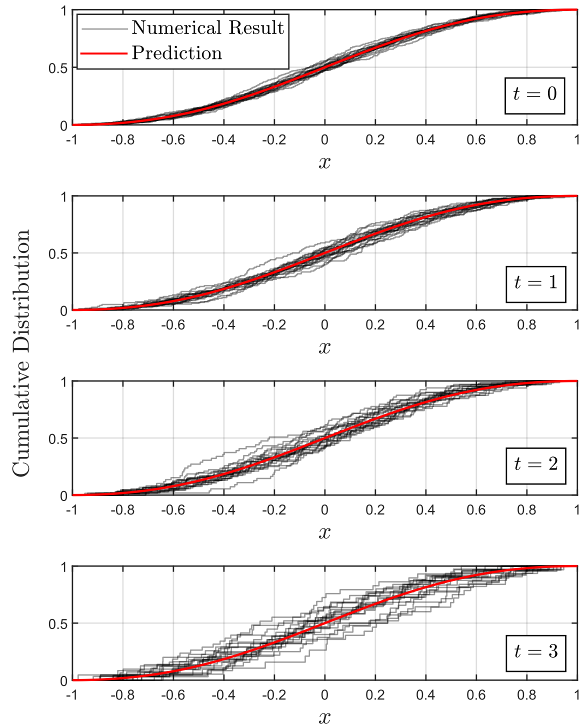

Fig. 3. The conditional cumulative distribution captured during the simulation. Each gray line denotes the cumulative histogram for a batch, i.e., the number of *surviving* trials having values less than x divided by the number of total surviving trials at time t. The red curves denote the predicted value $\Psi(x) = \int_{-1}^{x} q(u)du$. The measured distribution well matches the prediction, although the variance increases as the number of surviving samples decay with respect to time.

to validate the formula. From the result, it can be seen that the calculated analytical leakage rate well describes the numerical result. As discussed briefly in section III-C, while we only considered autonomous systems in this paper, for future work, we plan to develop a feedback strategy for a given set C to bound the leakage rate from above.

Appendix: Proof of Lemma 1

Proof. Define two stochastic processes as follows:

$$\begin{aligned} W_t' &= \frac{\partial_x h(x) \cdot \sigma(x)}{\|\partial_x h(x) \cdot \sigma(x)\|} W_t, \\ H_t &= h(x) + \partial_x h(x) \cdot \sigma(x) \cdot W_t \\ &= h(x) + \|\partial_x h(x) \cdot \sigma(x)\| \cdot W_t'. \end{aligned} \tag{46}$$

Then, W_t' is a standard one-dimensional Brownian motion and therefore the following holds [23, Theorem 3.7.1]:

$$\mathbb{P}\left[\min_{\tau \in [0,t]} W_\tau' < -\alpha\right] = 2\Phi\left(-\frac{\alpha}{\sqrt{t}}\right), \quad \forall \alpha, t \geq 0 \tag{47}$$

and

$$\mathbb{P}\left[\min_{\tau \in [0,t]} H_\tau < a t^{1/4}\right] = 2\Phi\left(-\frac{h(x) - a t^{1/4}}{\|\partial_x h(x) \cdot \sigma(x)\| \sqrt{t}}\right) \tag{48}$$

for any $a \in \mathbb{R}$, given sufficiently small t. Making use of the strictly increasing and convex property of the functions ϕ

and Φ for negative values with large enough magnitude, one can obtain

$$\begin{aligned}\left|\mathbb{P}\left[\min_{\tau\in[0,t]} H_\tau < at^{1/4}\right] - 2\Phi\left(-\frac{h'}{\sqrt{t}}\right)\right| \\ \leq 4\frac{|a'|}{t^{1/4}} \cdot \left|\phi\left(-\frac{h'}{\sqrt{t}} + \frac{|a'|}{t^{1/4}}\right)\right| = o(t^k), \quad \forall k \geq 0\end{aligned} \tag{49}$$

for t close to zero, where h' and a' are shorthands for $h(x)/\|\partial_x h(x) \cdot \sigma(x)\|$ and $a/\|\partial_x h(x) \cdot \sigma(x)\|$, respectively.

A well-established proof of strong convergence of the Euler-Maruyama method for simulating Itô diffusion processes [24] states that

$$\mathbb{E}|H_t - h(X_t)| = O(\sqrt{t}). \tag{50}$$

Applying Markov's inequality to this, with probability greater than $1 - t^{1/4}$,

$$|h(X_t) - H_t| \leq ct^{1/4} \tag{51}$$

for some constant $c > 0$.

Combining (49) and (51) completes the proof. □

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