2024 63rd IEEE Conference on Decision and Control (CDC) @ Milan, Italy | **WeA19.3**

Leakage Rate as a Measure of Continuous-Time Stochastic Set Invariance

Inkyu Jang, Minhyuk Yoon, and H. Jin Kim

Department of Aerospace Engineering and Automation and Systems Research Institute,
Seoul National University, Seoul, Korea

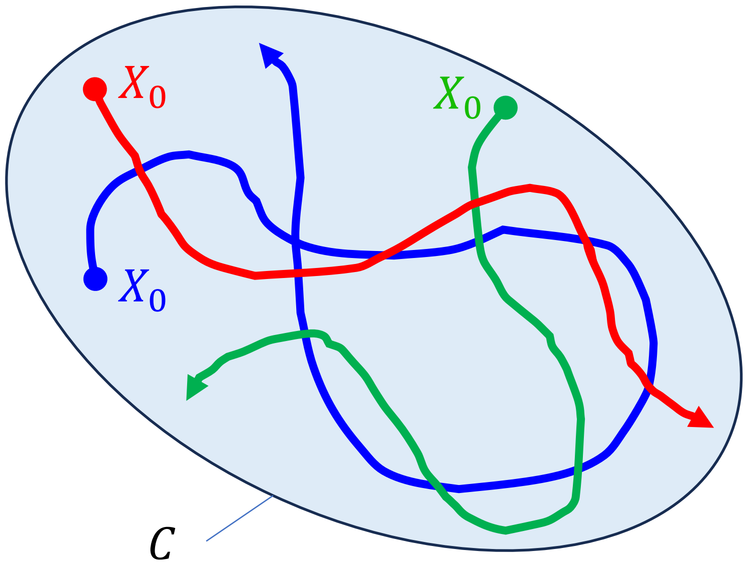
Motivation 1: Set Invariance and Safety

- Set invariance is a very important concept in control systems due to its connection to safety.
- A dynamical system can be regarded safe only if there exists a safe control invariant set.

Definition (Set Invariance)

A set C is (forward) invariant with respect to the dynamical system X_t if

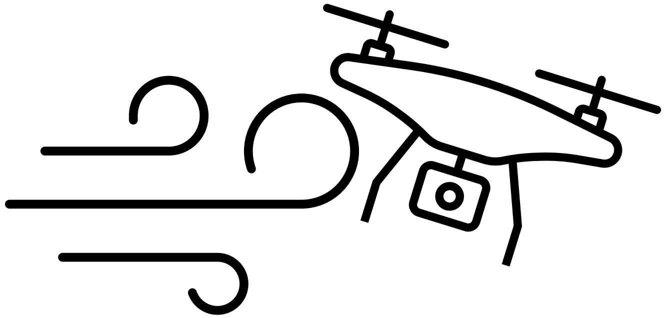
$$X_0 \in C \;\Rightarrow\; X_t \in C, \forall t \in [0, \infty)$$



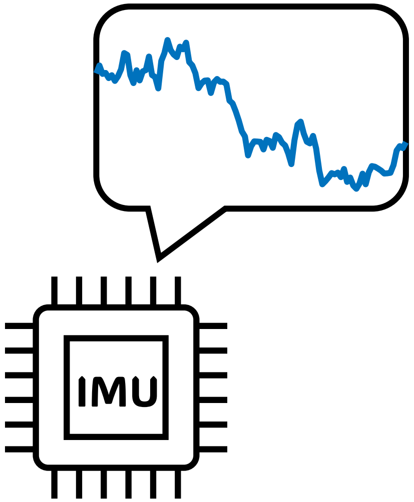
Motivation 2: Stochastic Systems are Ubiquitous

- Many systems in real-world are found to be stochastic.

Inherent stochasticity



External forces



Sensing noise

Itô Diffusion Processes

Definition (Itô Diffusion Process)

A (time-homogeneous) Ito diffusion is a stochastic process $\{X_t\}_{t\in[0,\infty)}$ satisfying the stochastic differential equation (SDE) in the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where W_t is the Wiener process and μ and σ are Lispchitz.

$$W_t = \lim_{t_{i+1}-t_i \searrow 0} \sum_{i=0}^{N-1} Z_i \cdot \sqrt{t_{i+1} - t_i}$$

Z_i are iid unit Gaussian,
$0 = t_0 < t_1 < \cdots < t_N = t$

- Itô diffusion process is a useful tool to describe continuous-time stochastic process with continuous trajectories.
- The drift term μ describes how the state will move *in average.*
- The diffusion term σ describes how *unpredictably* the state will change.

Probabilistic Set Invariance

- Probabilistic set invariance is the stochastic version of invariant sets [1], [2].

Definition (Probabilistic set invariance)

Given a time horizon $T \geq 0$, the set C is invariant with probability $1 - \epsilon$ if

$X_0 \in C$, then with probability $1 - \epsilon$, $X_t \in C$ for all $t \in [0, T]$

Q) Is this definition always useful?

[1] Kofman, Ernesto, José A. De Doná, and Maria M. Seron. "Probabilistic set invariance and ultimate boundedness." *Automatica* 48.10 (2012): 2670-2676.
[2] Gao, Yulong, Karl Henrik Johansson, and Lihua Xie. "Computing probabilistic controlled invariant sets." *IEEE Transactions on Automatic Control* 66.7 (2020): 3138-3151.

Probabilistic Invariance is Impractical for Itô Diffusion Processes

1. With bounded μ and nonzero σ, if $X_0 \in \partial C$, for any $T > 0$, X_t almost surely visits outside C at least once during $t \in [0, T]$. [1]

Reason (informal): Suppose $C = \{x : h(x) \geq 0\}$.

On ∂C, $\mathbb{E}[dh] \approx dt$, $\mathrm{Stdev}[dh] \approx \sqrt{dt}$. The diffusion term dominates dh.

The drift should be unbounded to compensate the large diffusion.

[1] Wang, Chuanzheng, et al. "Safety-critical control of stochastic systems using stochastic control barrier functions." *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE, 2021.

Probabilistic Invariance is Impractical for Itô Diffusion Processes

2. Considering the worst-case probability (w.r.t. the initial state) is overly conservative.

- The safety probability depends on the initial distribution of the state.

3. We want an invariance measure that does not depend on the time horizon T.

Leakage Rate: Definition

Definition (Leakage Rate)

Suppose the Itô diffusion process starts from an initial distribution X_0

$\Gamma(t) = \mathbb{P}\left[X_\tau \in C, \forall \tau \in [0,t]\right]$ is the *survival rate* of the system. $\Gamma(t)$ *decays* as time flows.

The leakage rate $\gamma(t)$ is defined as

$$\gamma(t) \coloneqq \lim_{\delta t \to 0} \frac{\mathbb{P}[\exists s \in [0, \delta t],\ X_{t+s} \notin C | X_\tau \in C, \forall \tau \in [0,t]]}{\delta t} = \lim_{\delta t \to 0} \frac{\Gamma(t) - \Gamma(t + \delta t)}{\delta t \cdot \Gamma(t)} = -\frac{\dot{\Gamma}(t)}{\Gamma(t)}$$

Leakage Rate: Interpretation

- $\gamma(t)$ measures the *decaying rate* of the survival rate and thus is a safety measure.

- Given the Markovian nature of Itô processes, Leakage rate depends only on the current survival-conditioned state density function $q(x,t)$.

  (i.e., the distribution of X_t given $X_\tau \in C, \forall \tau \in [0,t]$).

Problem

Given the dynamics and $q(x,t)$, accurately compute the leakage rate $\gamma(t)$ from the level set $C = \{x : h(x) \geq 0\}$

Main Result

Theorem (Main Result)

If there exists a continuous function $m_t: C \rightarrow \mathbb{R}_{\geq 0}$ such that $q(x,t) = m_t(x)h(x)$ almost everywhere on C,

then $\gamma(t)$ has a finite value

$$\gamma(t) = \int_{\partial C} \frac{m_t(x)\|\partial_x h(x) \cdot \sigma(x)\|^2}{2\|\partial_x h(x)\|} dS$$

$\int_{\partial C}(\cdots)dS$ is the surface integral over the boundary of C.

Implications of the Main Result (1/3)

Theorem (Main Result)

$$\gamma(t) = \int_{\partial C} \frac{m_t(x)\|\partial_x h(x)\cdot\sigma(x)\|^2}{2\|\partial_x h(x)\|}\, dS$$

- The leakage rate increases with bigger σ and m_t values on the boundary.
  - The set is *less invariant* with bigger diffusion (noise) term.
  - The set is *less invariant* with more chance of the state being near the boundary.
- $\gamma(t)$ remains invariant with respect to different representations of C.
  - It does not change with different h, as long as its super zero level set remains unchanged.

Implications of the Main Result (2/3)

Theorem (Main Result)

$$\gamma(t) = \int_{\partial C} \frac{m_t(x)\|\partial_x h(x) \cdot \sigma(x)\|^2}{2\|\partial_x h(x)\|} dS$$

- The leakage rate γ does not explicitly depend on the state distribution in the interior of C.
  - It only depends on how densely the state is concentrated on the boundary of C.
- The leakage rate does not explicitly depend on μ.
  - There is no way of *instantaneously reducing* γ, unless σ is directly controllable.

Implications of the Main Result (3/3)

Theorem (Main Result)

$$\gamma(t) = \int_{\partial C} \frac{m_t(x)\|\partial_x h(x) \cdot \sigma(x)\|^2}{2\|\partial_x h(x)\|} dS$$

- Finite $\gamma(t)$ implies that $q(x,t) = m_t(x)h(x)$ is zero on ∂C.
  - Bounded μ and nonzero σ will keep $q(x,t)$ zero on ∂C.
  - Otherwise, $\Gamma(t)$ will instantaneously drop to zero.

A Simple Numerical Example

- We evaluate the leakage rate of a simple Ornstein-Uhlenbeck process with one-dimensional X_t

$$dX_t = -X_t dt + dW_t$$

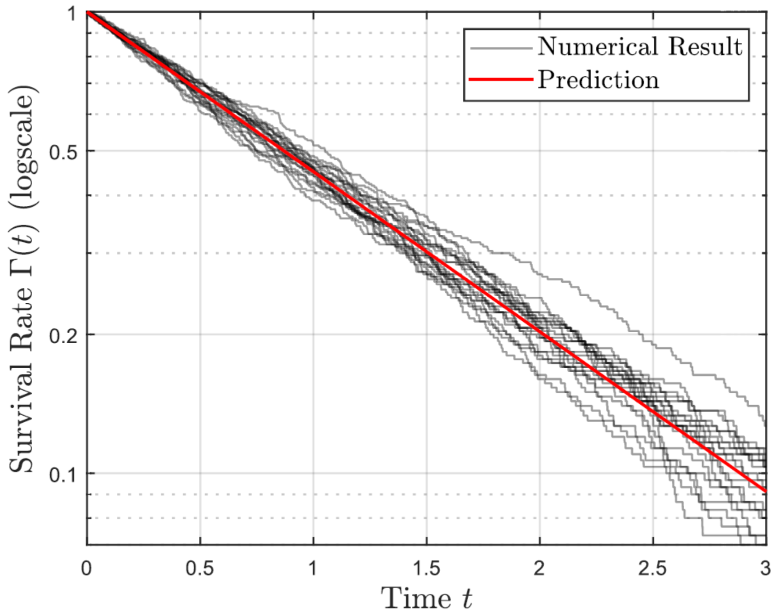
from the set $C = \{x: -1 \leq x \leq 1\}$.

- With $h(x) = 1 - x^2$,

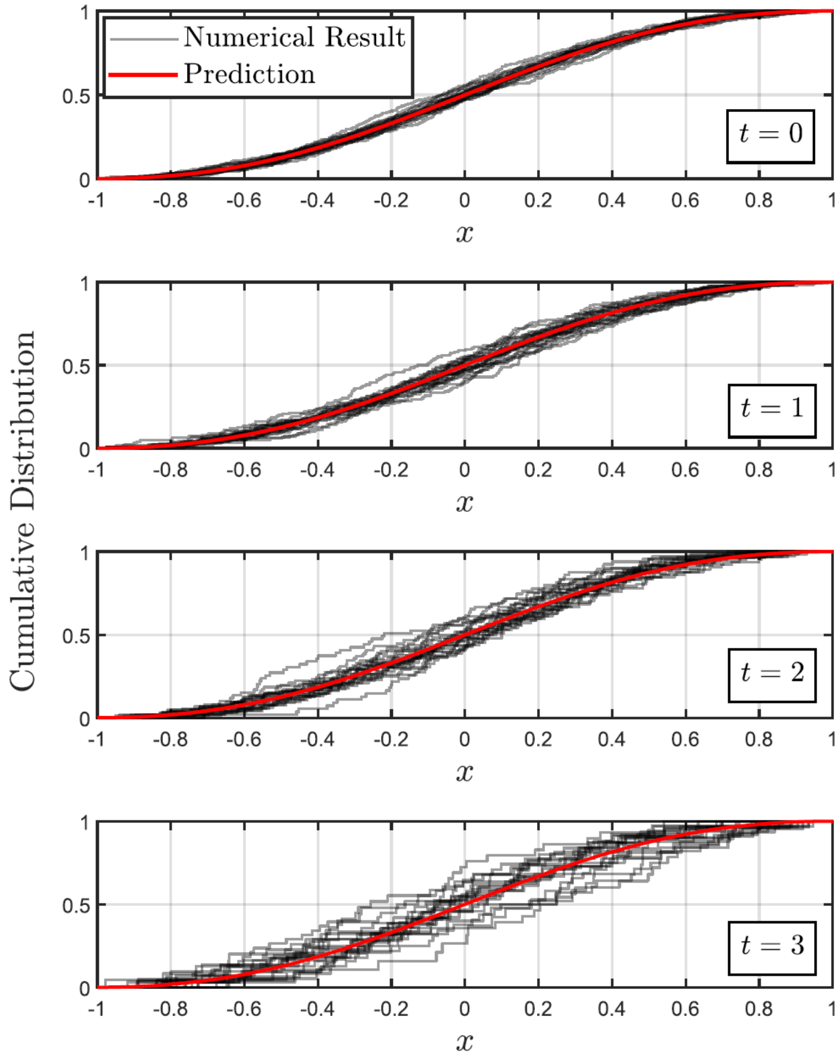
$\gamma < \infty$ if and only if $q(-1) = q(1) = 0$,

$$\gamma = \frac{m_t(-1) \cdot h'(-1)^2}{2h'(-1)} + \frac{m_t(1) \cdot h'(1)^2}{2h'(1)}$$
$$= \frac{q'(-1) - q'(1)}{2}$$

A Simple Numerical Example



Survival rate $\Gamma(t)$,
Monte-Carlo simulation result and prediction.



Survival-conditioned cumulative distribution on C,
Monte-Carlo simulation result and prediction.

Summary and Future Work

In this work, we have

- derived a surface integral formula that evaluates the leakage rate (decay rate of the safety probability) of an Itô diffusion process from a given level set.
- discovered that, with bounded drift and nonzero diffusion, the leakage rate is bounded only if the density function approaches zero near the boundary of the set almost surely.

Possible future studies include

- considering the more general case with control inputs.
- looking for connections between stochastic barrier-like functions.

Thank you!

Contact

Inkyu Jang
Department of Aerospace Engineering, Automation and Systems Research Institute
Seoul National University, Korea
janginkyu.larr@gmail.com

This work was supported in part by the Korea Research Institute for Defense Technology Planning and Advancement (KRIT) Grant funded by the Defense Acquisition Program Administration (DAPA) (No. KRIT-CT-23-003). The work of Inkyu Jang was supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (RS-2024-00407121).