Upper Bound on Escape Probability for Stochastic Control Barrier Functions

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***Abstract*— This paper introduces a new upper bound on escape probability of a stochastic system from a super zero level set of a zeroing-type stochastic control barrier function (SCBF). For both discrete-time and continuous-time cases, a time-varying probability bound is built by constructing a supermartingale and then applying Ville's inequality. While existing martingale-based probability bounds depend only on the expectation of change of the SCBF value, the proposed bound also takes into account the growth rate of its variance, resulting in enhanced tightness. The construction of the proposed bound does not require the SCBF value to be bounded, thus it is more suitable for tasks with large and non-compact safe sets. The validity and tightness of the proposed bound are checked using Monte-Carlo simulation experiments.**

***Index Terms*— Stochastic systems, Constrained control, Nonlinear systems, Markov processes**

I. INTRODUCTION

A. Motivation for Stochastic Control Barrier Functions

Safety is undeniably one of the most fundamental concerns in many control tasks. Formally verifying safety has therefore long attracted considerable attention from researchers in the related field. Safety in control systems engineering is usually defined as the existence of a control law that allows permanent satisfaction of a set of safety constraints, in other words, control invariance of the safe set. Although finding a control invariant set is known to be challenging for general nonlinear systems, many prior works have demonstrated promising results [1], for example, the Hamilton-Jacobi reachability analysis framework [2]–[4] which allows computation of the tightest possible invariant set, and control barrier functions (CBFs) [5]–[7]. These provide hard guarantees on set invariance given deterministic pre-known bounds on how the system will behave.

However, many real-world control systems are inherently stochastic, for example, robot control under disturbing forces or measurement uncertainty. In such cases, safety can no longer be checked deterministically nor be considered a binary (safe or unsafe) concept: Deterministic or almost-sure safety guarantee would typically require impractical assumptions, for example, unbounded input [8]. Instead, we are interested in estimating at which probability the system will escape a set within the prescribed time interval given the initial condition, i.e., the *escape probability* [9]–[12].

Regulating the escape probability and achieving best performance are the two most major but possibly conflicting goals of control synthesis for a stochastic system. A controller prioritizing performance over safety is likely to be aggressive and may put the system at a risk, and the safest possible controller would be the one that does nothing other than pushing the system to the safest spot. Stochastic CBF (SCBF) is a variant of CBF built upon this motivation [8], which is capable of generating input constraints that well balance between the two extremes, yielding a well-performing (yet not too conservative) control strategy.

In this paper, we use the notion *survival rate* and escape probability interchangeably for the sake of notational brevity. The survival rate $\Gamma(t)$ denotes at which probability the system will stay within the safe set throughout the time window $[0, t]$ and the escape probability is equal to $1 - \Gamma(t)$. Intuitively, $\Gamma(t)$ should be a monotonically decaying function and should depend on the initial state distribution and the dynamics.

B. Martingale-Based Escape Probability Bounds

While it is empirically obvious that a valid SCBF does regulate the escape probability, its actual upper bound does not generally relate to the SCBF parameters in a straightforward manner. Therefore, considerable research effort was recently devoted to derive a sufficiently tight bound of escape probability given a valid SCBF. A common procedure used in many works is to make use of one of the martingale inequalities, for example, Ville's inequality [13].

Definition 1 (Martingale)**.** Let $Y_t \in \mathbb{R}$ be a stochastic process indexed by time $t \geq 0$, where t can be either discrete or continuous. Let $Y_{[0,s]}$ be the trajectory information of Y throughout the interval $[0, s]$. Then, Y_t is a martingale if $\mathbb{E}[Y_t|Y_{[0,s]}] = Y_s$, supermartingale if $\mathbb{E}[Y_t|Y_{[0,s]}] \leq Y_s$, submartingale if $\mathbb{E}[Y_t|Y_{[0,s]}] \geq Y_s$, for all $s \leq t$.

Theorem 1 (Ville's Inequality [13])**.** *Let $Y_t \in \mathbb{R}$ be a nonnegative supermartingale. Then for any $\lambda > 0$,*

$$\mathbb{P}\left[\sup_{0\leq s\leq t} Y_s > \lambda\right] \leq \frac{\mathbb{E}Y_0}{\lambda}. \tag{1}$$

For discrete-time systems, works such as [11] suggested an exponentially decaying lower bound for the survival rate in the form $\Gamma(t) \geq \Gamma_0\gamma^t$ for appropriate constants Γ_0 and γ. For continuous-time systems, linear bounds $\Gamma(t) \geq \Gamma_0 - \gamma t$ [14] or

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exponentially decaying bounds $\Gamma(t) \geq \Gamma_0 e^{-\gamma t}$ [10], [15] were derived. The methods mentioned here all require the SCBF value to be bounded in the safe set. This might pose hurdles in SCBF synthesis and introduce additional conservativeness especially if the safe set is not compact, for example, collision avoidance task for mobile robots.

As its appearance suggests, Ville's inequality is a result of Markov's inequality[1] and it depends only on the expectation of Y_0 and not on any additional information on how Y_t is distributed around the expectation, for example, its variance. In other words, the inequality holds for Y_t with arbitrary variance. This is a big strength, because the derived probability bound does not depend on the characteristics of the uncertainty (or noise) and therefore generalizes to a wider class of systems. On the other hand, this also is the main cause of the often overconservativeness in the abovementioned works, especially with small variances.

The authors of [16] precisely pointed out this issue and suggested a comparison-lemma-like theorem that gives a significantly tighter bound. However, evaluation of this bound requires Monte-Carlo simulation of a stochastic process, and the method only applies to continuous-time Itô diffusion processes with deterministic initial condition.

C. Summary of Contributions of the Paper

In this paper, we derive new escape probability bounds for both discrete-time and continuous-time SCBFs based on Ville's inequality. While not losing the abovementioned strength of martingale-based methods, the proposed bounds improve existing works in the following three perspectives.

- The proposed probability bound explicitly depends on the magnitude of uncertainty of the dynamics. In the extreme case where the system becomes deterministic, the proposed method yields $\Gamma(t) = 1$.
- The bound takes into account the non-deterministic initial condition. The probability depends on the mean and variance of the initial SCBF value.
- Empirically, the proposed bound exhibits less conservativeness compared to existing works.

The first two advantages come from the *squaring* technique we used in deriving the probability bound. This is similar to the idea to derive Chebyshev's inequality[2] from Markov's inequality. This squaring technique lets the variance information directly appear in the supermartingale conditions, making the probability bound explicitly depend on the variance. Since the bounds depend on the variance, we accordingly introduce a modification to the definition of SCBFs, in which not only their expectation but also the variance of drift of SCBF values is bounded. To verify the last item, we compare our discrete-time bound with [11] and the continuous-time probability bound with [10], [14]. We also present how the proposed bound compare with Monte-Carlo simulation results.

[1]For a nonnegative random variable X with finite expectation, $\mathbb{P}[X \geq \lambda] \leq \mathbb{E}X/\lambda$ for all $\lambda > 0$.

[2]For a random variable $X \in \mathbb{R}$ with finite expectation μ and variance $\sigma^2 > 0$, $\mathbb{P}[|X-\mu| \geq k\sigma] \leq 1/k^2$ for any $k > 1$. The proof starts by applying Markov's inequality to the nonnegative random variable $|X-\mu|^2$.

II. Discrete-Time SCBF

A. Definition

Consider the discrete-time stochastic dynamics model

$$X_{t+1} \sim F(X_t, u_t) \tag{2}$$

where $X_t \in \mathbb{X}$ and $u_t \in U$ are the state and input at time $t \in \{0, 1, \cdots\}$, respectively. The map F describes the probability distribution of the next state X_{t+1} given the current state and input pair (X_t, u_t). Discrete-time SCBF (DTSCBF) is a concept inspired by discrete-time CBFs [17]. We use the following definition in this paper.

Definition 2 (Discrete-Time SCBF)**.** Let $a \in (0,1)$ and $b \geq 0$ be constants. A continuous function $h : \mathbb{X} \to \mathbb{R}$ is an (a,b)-DTSCBF if, for every $x \in C = \{x : h(x) \geq 0\}$, there exists an input u such that

$$\mathbb{E}_{X\sim F(x,u)}[h(X)] \geq ah(x) + b, \tag{3}$$
$$\mathrm{Var}_{X\sim F(x,u)}[h(X)] \leq 1 \tag{4}$$

For every $x \in \mathbb{R}^n$, denote the set of control inputs that satisfies (3) and (4) by $U_{\mathrm{dtscbf}}(x)$, i.e.,

$$U_{\mathrm{dtscbf}}(x) := \left\{ u \in U : \begin{array}{l} \mathbb{E}_{X\sim F(x,u)}[h(X)] \geq ah(x) + b, \\ \mathrm{Var}_{X\sim F(x,u)}[h(X)] \leq 1 \end{array} \right\} \tag{5}$$

The parameters a, b determine how aggressively in average the DTSCBF will push the state into the set C near its boundary ∂C: One can easily expect that the escape probability increases with smaller a or b values. On the other hand, the variance condition (4), which is original in this paper, measures how *unpredictably* the h value will change. The introduction of this requirement can be regarded natural since this unpredictability obviously directly affects the escape probability. The upper bound 1 is not a restrictive constraint: Whenever the upper bound of the variance does not match 1, we can always re-scale h by a positive real number to meet the unit upper bound.

B. Exit Probability Upper Bound

Now, given the definition, we will derive a lower bound of the survival rate

$$\Gamma(t) := \mathbb{P}_{X_0}\left[\min_{0\leq s\leq t} h(X_s) \geq 0\right] \tag{6}$$

using Ville's inequality given the initial state distribution X_0. We begin with the following lemma.

Lemma 1 (Comparison in Exit Probability, Discrete-Time)**.** *Let h be a (a,b)-DTSCBF. For a set of parameters $\beta \leq b$, and $\delta \geq 0$, define a discrete-time random process L_t as*

$$L_0 = h(X_0) - \delta,$$
$$L_{t+1} = \begin{cases} aL_t + \beta + h(X_{t+1}) - \mathbb{E}h(X_{t+1}) & \text{if } h(X_t) \geq 0, \\ aL_t + \beta & \text{else} \end{cases} \tag{7}$$

and assume $u_t \in U_{\mathrm{dtscbf}}(X_t)$ for all t. Then,

$$\mathbb{P}\left[\min_{0\leq s\leq t} h(X_s) < 0\right] \leq \mathbb{P}\left[\min_{0\leq s\leq t} L_s < 0\right]. \tag{8}$$

Proof. We will show that if $\min_{0\le s\le t} h(X_s) \ge 0$ (i.e., the first case of the update rule in (7) applies), then

$$L_s \ge h(X_s) \quad \forall s \in [0,t], \tag{9}$$

from which (8) directly follows. Define a random process Δ_s ($0 \le s \le t$) as

$$\Delta_s := \frac{h(X_s) - L_s}{a^s}, \tag{10}$$

so that $\Delta_s \ge 0$ if and only if $h(X_s) \ge L_s$. Firstly, $\Delta_0 = \delta \ge 0$. And the increment of Δ at time $s \in [0,t)$ is

$$\begin{aligned}\Delta_{s+1} - \Delta_s &= \frac{h(X_{s+1}) - L_{s+1} - ah(X_s) + aL_s}{a^{s+1}} \\ &= \frac{\mathbb{E}h(X_{s+1}) - aL_s - \beta}{a^{s+1}} \\ &\ge \frac{ah(X_s) + b - aL_s - \beta}{a^{s+1}} \\ &= \frac{a\Delta_s + (b-\beta)}{a^{s+1}}\end{aligned} \tag{11}$$

which is nonnegative given $\Delta_s \ge 0$. Applying mathematical induction on $s \in [0,t)$ completes the proof. □

Now, Ville's inequality gives the following theorem.

Theorem 2 (Survival Rate Lower Bound, Discrete-Time)**.** *If $h : \mathbb{X} \to \mathbb{R}$ is an (a,b)-DTSCBF and u_t is picked from the set $U_{\mathrm{dtscbf}}(X_t)$, then*

$$\begin{aligned}&\mathbb{P}\left[h(X_s) \ge 0,\ \forall s \in [0,t]\right] \\ &\ge \begin{cases}\left(1 - \frac{\mathrm{Var}(h(X_0))}{v^2}\right)\gamma^t & \text{if } \mathbb{E}h(X_0) \ge v \\ \left(1 - \frac{\mathbb{E}\left[(h(X_0)-v)^2\right]}{v^2}\right)\gamma^t & \text{else}\end{cases}\end{aligned} \tag{12}$$

for any pair of $v \ge (1+b^2)/2b$, $\gamma \in (a^2, 1]$ such that

$$v \le \frac{b}{1-a} \ \wedge\ \gamma = \max\left\{a, 1 - \frac{1}{v^2}\right\} \tag{13}$$

or

$$v \ge \frac{b}{1-a} \ \wedge\ (ab - va + v\gamma)^2 - (1 - 2vb + b^2)(a^2 - \gamma) = 0. \tag{14}$$

Proof. This proof is structured as follows. Firstly, we set up a supermartingale candidate, with which Ville's inequality gives the probability bound (12). Then, we derive the condition so that the candidate becomes a supermartingale.

For fixed $\beta \in [0,b]$, $\delta \ge 0$, $v \ge 0$, $\gamma \in [a^2, 1)$ and time horizon t, let

$$Y_s = \frac{(L_s - v)^2 - v^2}{\gamma^s} + \frac{v^2}{\gamma^t} \tag{15}$$

be a discrete time stochastic process defined on the time interval $s \in [0,t]$. Observe that $Y_s > v^2\gamma^{-t}$ whenever $L_s < 0$. Thus, if Y_s is a supermartingale, then combining Lemma 1 and Ville's inequality gives

$$\begin{aligned}\mathbb{P}\left[\min_{0\le s\le t} h(X_s) < 0\right] &\le \mathbb{P}\left[\min_{0\le s\le t} L_s < 0\right] \\ &\le \mathbb{P}\left[\max_{0\le s\le t} Y_s < v^2\gamma^{-t}\right] \\ &\le \frac{\mathbb{E}Y_0}{v^2}\gamma^t \\ &= 1 - \left(1 - \frac{\mathbb{E}\left[(h(X_0)-v-\delta)^2\right]}{v^2}\right)\gamma^t.\end{aligned} \tag{16}$$

Letting $\delta = \max\{\mathbb{E}h(X_0) - v, 0\}$ results in (12).

Now, let us examine for which choice of β, v and γ (15) becomes a supermartingale. For that, suppose $h(X_s) \ge 0$ and observe the behavior of the quantity

$$\begin{aligned}\Delta Y_s &= (Y_{s+1} - Y_s)\gamma^{s+1} \\ &= L_{s+1}^2 - \gamma L_s^2 - 2vL_{s+1} + 2v\gamma L_s\end{aligned} \tag{17}$$

If $\mathbb{E}[\Delta Y_s | X_{[0,s]}, L_s] \le 0$ for all $L_s \in \mathbb{R}$ and $s \in [0,t]$, then Y_s surely is a supermartingale. This conditional expectation is

$$\begin{aligned}\mathbb{E}[\Delta Y_s] &= \mathbb{E}\left[(aL_s + \beta + h(X_{s+1}) - \mathbb{E}h(X_{s+1}))^2\right] \\ &\quad - \gamma L_s^2 - 2v(al + \beta) + 2v\gamma L_s \\ &= (a^2 - \gamma)L_s^2 + 2(a\beta - va + v\gamma)L_s \\ &\quad + \mathrm{Var}(h(X_s)) - 2v\beta + \beta^2\end{aligned} \tag{18}$$

for the case $h(X_s) \ge 0$ and

$$\begin{aligned}\mathbb{E}[\Delta Y_s] &= (aL_s + \beta)^2 - \gamma L_s^2 - 2v(al + \beta) + 2v\gamma L_s \\ &= (a^2 - \gamma)L_s^2 + 2(a\beta - va + v\gamma)L_s - 2v\beta + \beta^2\end{aligned} \tag{19}$$

otherwise, both being upper bounded by

$$\mathbb{E}[\Delta Y_s] \le (a^2 - \gamma)L_s^2 + 2(a\beta - va + v\gamma)L_s + 1 - 2v\beta + \beta^2. \tag{20}$$

Here, all the expectations and the variance in (18), (19), and (20) are conditioned on the previous trajectory $X_{[0,s]}$ (and therefore L_s also). The key of introducing variance information here is the construction of (15), which we called the squaring technique in Section I-C. The bound (20) is nonpositive for all $L_s \in \mathbb{R}$ if and only if the three conditions are met:

$$a^2 - \gamma \le 0, \tag{21}$$

$$1 - 2v\beta + \beta^2 \le 0, \tag{22}$$

$$(a\beta - va + v\gamma)^2 + (\gamma - a^2)(1 - 2v\beta + \beta^2) \le 0. \tag{23}$$

That is, any choice of β, v, γ satisfying the above conditions will make Y_s a supermartingale. The first requirement (21) is satisfied with strict inequality $<$, which is already an assumption. For β to satisfy (22) and $\beta \le b$ simultaneously, it should lie between the bounds

$$v - \sqrt{v^2 - 1} \le \beta \le \min\left\{b, v + \sqrt{v^2 - 1}\right\}, \tag{24}$$

which has a real solution if $v \ge \frac{b^2+1}{2b}$. Subject to this, we

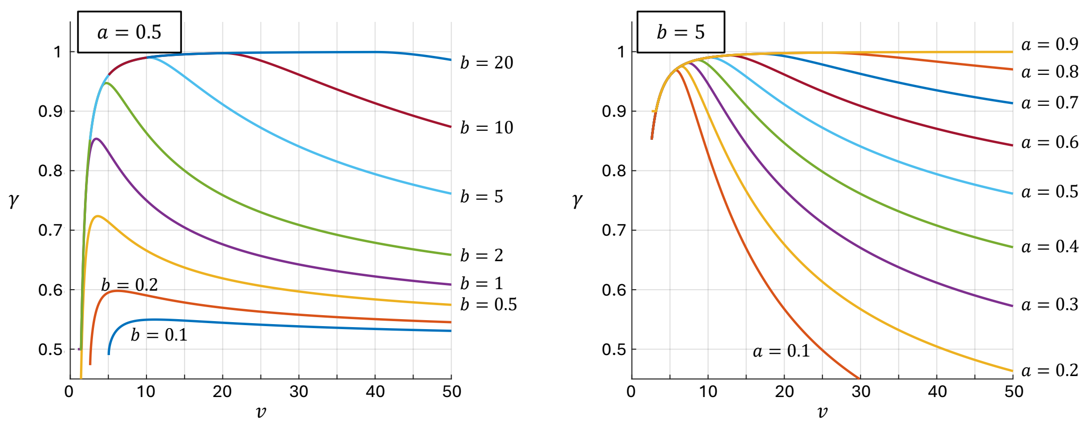

Fig. 1. The plot of γ values as in Theorem 2 (discrete-time version) for many pairs of a and b. The γ values close to 1 denote smaller decay of the survival probability $\Gamma(t)$. It can be clearly seen that the behavior follows the intuition that bigger a and b values result in higher chance of survival.

choose β that minimizes the left side of (23).

$$\beta = \underset{(24)}{\arg\min}\ (a\beta - va + v\gamma)^2 + (\gamma - a^2)(1 - 2v\beta + \beta^2)$$

$$= \begin{cases} v - \sqrt{v^2-1} & \text{(case A)} \\ v(1-a) & \text{(case B)} \\ b & \text{(case C)} \\ v + \sqrt{v^2+1} & \text{(case D)} \end{cases} \tag{25}$$

This QP has four solution candidates.

Case A: It applies when the lower bound $\beta = v - \sqrt{v^2-1}$ is the optimal solution. It happens if $v(1-a) < v - \sqrt{v^2-1}$, i.e., $v < 1/\sqrt{1-a^2}$. Here, $\gamma = a\sqrt{v^2-1}/v$ is the only solution for the inequality (23). However, such γ is always smaller than a^2, which contradicts with (21). Thus, case A is void.

Case B: B applies if the unconstrained solution $\beta = v(1-a)$ satisfies (24). This is equivalent to $v \leq b/(1-a)$. In this case, (23) can be written

$$\begin{aligned} 0 &\geq (a\beta - va + v\gamma)^2 + (\gamma - a^2)(1 - 2v\beta + \beta^2) \\ &= (\gamma - a)(1 - v^2 + \gamma v^2), \end{aligned} \tag{26}$$

which leads to (13).

Case C: C applies if $b \leq v + \sqrt{v^2-1}$ and $v(1-a) > b$. It is always true that $v + \sqrt{v^2-1} > v(1-a)$ given $a > 0$ and $v \geq 1$, so this reduces to the condition $v(1-a) > b$, i.e., $v \geq b/(1-a)$. Substituting β gives (14).

Case D: Lastly, case D applies if $b > v + \sqrt{v^2-1}$ and $v(1-a) \geq v + \sqrt{v^2-1}$. Similarly to case A, the only solution for (23) is $\gamma = -a\sqrt{v^2-1}/v$ which is negative, contradicting with (21). Thus, case D is void. □

Since (13) is a quadratic equation with respect to γ, we can find a closed-form expression for the root γ, should one exist, as a function of v. That is, combining with (14), γ can be written in a piecewise smooth function of v whose plots for various (a, b) pairs are shown in Fig. 1. The γ values close to 1 denote slower decay in survival probability. It can be clearly seen in Fig. 1 that for fixed v, γ is monotonically increasing with respect to both a and b, which precisely matches the intuition from Section II-A. It can also be found that γ value asymptotically approaches 1 with growing v and b. This means that Theorem 2 says $\Gamma(t)$ converges to 1 with sufficiently small uncertainty, i.e., nearly deterministic systems.

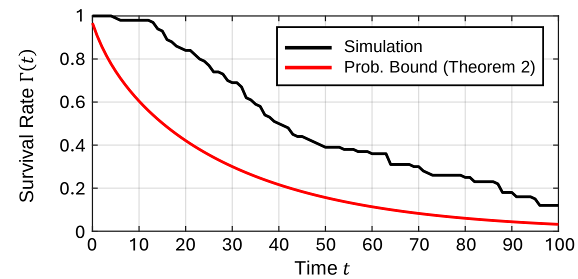

Fig. 2. Monte Carlo simulation result for the numerical example system (27). Whenever the indoor temperature hits 25°C, the simulation terminates. The simulated survival rate is an estimate of the true value of $\Gamma(t)$ given by number of surviving trajectories divided by the number of total simulations, which is 100 in this example. It can be seen that Theorem 2 provides a lower bound of $\Gamma(t)$ that is not overly conservative. Prob.=Probability.

C. Numerical Example

For a numerical example for the discrete-time case, we consider an air conditioning task which is modeled as follows:

$$X_{t+1} = \begin{bmatrix} 0.9 + 0.05(1-u_t) & 0.05 \\ 0 & 0.99 \end{bmatrix} X_t + \begin{bmatrix} 0.25u_t + 0.15w_t \\ 0.3 + 0.2z_t \end{bmatrix} \tag{27}$$

where the first and second components of $X_t \in \mathbb{X} = \mathbb{R}^2$ respectively denote the indoor and outdoor temperatures in °C, w_t and z_t are independent and identically distributed (i.i.d.) one-dimensional unit Gaussians, and $u_t \in [0, 1]$ is the input with $u_t = 0$ denoting the air conditioner off, $u_t = 1$ running at full throttle. The goal of this task is to keep the indoor temperature below 25°C, where the initial condition begins at $X_0 = [24, x_0]^\top$ where x_0 is uniformly distributed between 30 and 40. For this task, we use

$$h(x) = \min\{100 - 4x_1, 160 - 4x_2\}, \tag{28}$$

where x_1 and x_2 are the first and second components of x, respectively. One can find that this h is a $(0.9, 0.4)$-DTSCBF (see the appendix). Leveraging the convexity of h, we construct a lower estimate of $\mathbb{E}h(X_{t+1})$ as $\mathbb{E}h(X_{t+1}) \geq h(\mathbb{E}[X_{t+1}])$, and the input u_t that satisfies $h(\mathbb{E}[X_{t+1}]) = ah(X_t) + b$ was used in the simulation. Monte Carlo simulation results using the numerical values are summarized in Fig. 2.

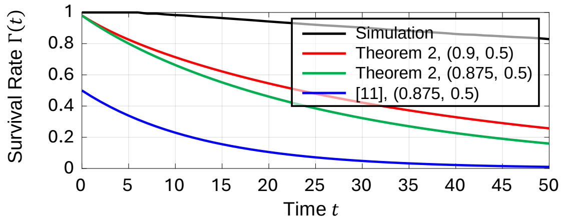

Fig. 3. Comparison result with [11]. The black curve named simulation denote the true value of $\mathbf{\Gamma}(t)$ estimated by the Monte Carlo simulation with 10000 samples. The numbers inside the brackets in the legend denote the parameter $\boldsymbol{a}$ and $\boldsymbol{b}$ values of the DTSCBF used. The proposed method (Theorem 2) exhibits less conservativeness compared to [11].

It can be seen that Theorem 2 provides a decent lower bound for the survival rate $\Gamma(t)$.

D. Comparison with Prior Work

We conclude this section by comparing with [11] using the simple stochastic system

$$X_{t+1} = 0.9X_t + 0.5 + d_t \tag{29}$$

where $X_t \in \mathbb{X} = \mathbb{R}$ is the one-dimensional state, and the disturbance term d_t takes value 1 with probability $1/2$ and -1 with probability $1/2$. Suppose that the initial condition is $X_0 = 5$ with probability 1 and let the DTSCBF candidate be $h(x) = x$. It is very straightforward to find that h is a $(0.9, 0.5)$-DTSCBF. Since [11] requires h to be bounded from above, a modified DTSCBF $h(x) = \min\{x, M\}$, where $M = 10$, is also tested. The new h is a $(0.875, 0.5)$-DTSCBF. Three survival rate lower bounds, for original and bounded h-s using Theorem 2 and for bounded h using [11], along with Monte Carlo estimation of the actual survival rate, is plotted in Fig. 3. Due to the conservative nature of Markov inequality variants, both methods exhibit non-negligible conservativeness. Nevertheless, it can be clearly seen that the proposed method gives a significantly tighter bound.

III. Continuous-Time SCBF

A. Definition

In this section, using the similar technique as the discrete-time case, we derive a bound on exit probability using Ville's inequality for SCBFs for continuous-time stochastic systems driven by the Itô stochastic differential equation (SDE) as follows:

$$dX_t = f(X_t, u_t)dt + \sigma(X_t, u_t)dW_t, \tag{30}$$

where $X_t \in \mathbb{R}^n$ as a random variable is the state at time $t \geq 0$, $u_t \in U \subseteq \mathbb{R}^m$ is the input, W_t is the m_w-dimensional standard Brownian motion. The continuous function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ describes how *in average* the system will drift, and $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m_w}$ is the *diffusion* term which measures the magnitude of uncertainty associated with the system's evolution.

Before defining continuous-time SCBF, we firstly introduce the concept of infinitesimal generator, which is the stochastic analogue of Lie derivative in deterministic systems.

Definition 3 (Infinitesimal Generator)**.** Given the dynamics (30) and an input $u \in U$, the infinitesimal generator $\mathcal{A}\zeta(x, u)$ of a function $\zeta : \mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\mathcal{A}\zeta(x, u) := \lim_{s\searrow 0} \frac{\mathbb{E}\left[\zeta(X_{t+s})\right] - \zeta(x)}{s}, \tag{31}$$

where the expectation is conditioned on $X_t = x$ and $u_\tau = u$ for all $\tau \in [t, t+s]$.

Definition 4 (Continuous-Time SCBF)**.** Let a and b be positive real numbers. A twice continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a (a, b)-CTSCBF if for every $x \in C := \{x : h(x) \geq 0\}$, there exists a feedback $u \in U$ such that

$$\mathcal{A}h(x, u) \geq -ah(x) + b, \tag{32}$$

$$\|\partial_x h(x)\sigma(x, u)\|_2 \leq 1. \tag{33}$$

The infinitesimal generator $h(x, u)$ can be evaluated using Itô's lemma as follows:

$$\mathcal{A}h(x, u) = \partial_x h(x)f(x, u) + \frac{1}{2}\mathrm{tr}\left(\sigma(x, u)^\top \partial_{xx}h(x)\sigma(x, u)\right), \tag{34}$$

where $\partial_x h(x) \in \mathbb{R}^{1\times n}$ and $\partial_{xx}h(x) \in \mathbb{R}^{n\times n}$ are the gradient and Hessian of h at x, respectively, and tr is the trace operator. For every state point x, we write the set of control inputs that satisfy the CTSCBF constraints as

$$U_{\mathrm{ctscbf}}(x) := \left\{u \in U : \begin{matrix} \mathcal{A}h(x, u) \geq -ah(x) + b, \\ \|\partial_x h(x)\sigma(x, u)\|_2 \leq 1 \end{matrix}\right\}. \tag{35}$$

Unlike the discrete-time case, the drift term σ is usually not controllable by the input u and is a function of x only. It is worth mentioning that if σ is a function of x only and f is linear with respect to the input u (i.e., input affine system), then the set (35) will be the intersection of U and an affine halfspace in $\mathbb{R}^m$. This allows one to construct a QP-based safety filter, namely (S)CBF-QP [18]. On the other hand, similar to DTSCBF, the diffusion regulation term (33), which regulates how severely the h value will diffuse, is not restrictive since h can be rescaled to meet the requirement. The parameters a and b determine the conservativeness of CTSCBF: one can expect smaller a and bigger b values would result in lower chance of escape.

B. Exit Probability Upper Bound

To derive the exit probability upper bound for CTSCBFs, we begin with the continuous-time version of Lemma 1. In this section, $B(x, u) \in \mathbb{R}^{1\times m_w}$ is a shorthand notation for $\partial_x h(x)\sigma(x, u)$.

Lemma 2 (Comparison in Exit Probability, Continuous-Time)**.** *Let h be an (a, b)-CTSCBF. For a set of parameters $\beta \leq b$, and $\delta \geq 0$, define the random process L_t by*

$$L_0 = h(X_0) - \delta,$$

$$dL_t = \begin{cases} (-aL_t + \beta)dt + B(X_t, u_t)dW_t & \text{if } h(X_t) \geq 0 \\ (-aL_t + \beta)dt & \text{else} \end{cases} \tag{36}$$

and assume $u_t \in U_{\mathrm{ctscbf}}(X_t)$ for all t. Then,

$$\mathbb{P}\left[\min_{0\leq s\leq t} h(X_s) < 0\right] \leq \mathbb{P}\left[\min_{0\leq s\leq t} L_s < 0\right]. \tag{37}$$

Proof. The proof is very similar to that of Lemma 1. We will show $h(X_s) \geq L_s$ a.s. for all $s \in [0,t]$ if $\min_{\tau\in[0,t]} h(X_\tau) \geq 0$. Assume nonnegativity of $h(X_s)$ throughout the interval $s \in [0,t]$, and let the stochastic process Δ_s ($s \in [0,t]$) defined by

$$\Delta_s := (h(X_s) - L_s)e^{as}. \tag{38}$$

Itô's lemma says Δ_s follows the SDE

$$\begin{aligned}\Delta_0 &= \delta,\\ d\Delta_s &= (\mathcal{A}h(X_s,u_s) + ah(X_s) - \beta)e^{as}ds.\end{aligned} \tag{39}$$

Since h is an (a,b)-CTSCBF, the quantity inside the bracket $(\cdots)$ is deterministically nonnegative

$$(\cdots) \geq -ah(X_s) + b + ah(X_s) - \beta = b - \beta \geq 0, \tag{40}$$

and since $\Delta_0 = \delta \geq 0$, we conclude that $\Delta_s \geq 0$, for all $s \in [0,t]$. □

With the help of this lemma, we obtain the following probability bound.

Theorem 3 (Survival Rate Lower Bound, Continuous-Time)**.** *If $h : \mathbb{R}^n \to \mathbb{R}$ is an (a,b)-CTSCBF and u_t is picked from $U_{\mathrm{ctscbf}}(X_t)$, then*

$$\begin{aligned}&\mathbb{P}\left[h(X_s) \geq 0,\ \forall s \in [0,t]\right]\\ &\geq \begin{cases}\left(1 - \frac{\mathrm{Var}(h(X_0))}{v^2}\right)e^{-\gamma t} & \textit{if } \mathbb{E}h(X_0) \geq v\\ \left(1 - \frac{\mathbb{E}\left[(h(X_0)-v)^2\right]}{v^2}\right)e^{-\gamma t} & \textit{else}\end{cases}\end{aligned} \tag{41}$$

for any pair of $v \geq b/2$, $\gamma \leq 2a$ such that

$$v > \frac{b}{a}\ \wedge\ ((a-\gamma)v + b)^2 + (2a-\gamma)(1-2bv) = 0 \tag{42}$$

or

$$\frac{1}{\sqrt{2a}} \leq v \leq \frac{b}{a}\ \wedge\ \gamma = \max\left\{2a, \frac{1}{v^2}\right\}. \tag{43}$$

Proof. The structure of the proof is similar to that of Theorem 2. We firstly construct a supermartingale candidate and then derive the conditions for the candidate to be a supermartingale.

For fixed $\beta \in [0,b]$, $\delta \geq 0$, $v \geq 0$, $\gamma \in [0,2a]$ and time horizon t, define L_s as Lemma 2 and let

$$Y_s = (L_s - v)^2 e^{\gamma s} - v^2 e^{\gamma s} + v^2 e^{\gamma t} \tag{44}$$

be a continuous-time stochastic process defined on $s \in [0,t]$. Similarly to Theorem 2, $Y_s > v^2 e^{\gamma t}$ whenever L_s becomes negative. We aim to search for the condition such that Y_s becomes a supermartingale, then Ville's inequality gives

$$\begin{aligned}\mathbb{P}\left[\min_{0\leq s\leq t} h(X_s) < 0\right] &\leq \mathbb{P}\left[\min_{0\leq s\leq t} L_s < 0\right]\\ &\leq \mathbb{P}\left[\max_{0\leq s\leq t} Y_s > v^2 e^{\gamma t}\right]\\ &\leq \frac{\mathbb{E}Y_0}{v^2}e^{-\gamma t}\\ &= 1 - \left(1 - \frac{\mathbb{E}\left[(h(X_0)-v-\delta)^2\right]}{v^2}\right)e^{-\gamma t}\end{aligned} \tag{45}$$

for any $\delta \geq 0$. Letting $\delta = \max\{\mathbb{E}h(X_0) - v, 0\}$ will give the probability bound (41).

According to Itô's lemma, Y_s follows the SDE

$$\begin{aligned}&dY_s =\\ &e^{\gamma s}\cdot[(\gamma-2a)L_s^2 + 2((a-\gamma)v+\beta)L_s + \hat{B}_s\hat{B}_s^\top - 2\beta v]ds\\ &+ e^{\gamma s}\cdot 2(L_s - v)\hat{B}_s dW_s,\end{aligned} \tag{46}$$

where $\hat{B}_s$ takes the value $B(X_s,u_s)$ if $h(X_s) \geq 0$, and 0 otherwise. This Y_s is a supermartingale if the quantity within the bracket $[\cdots]$ is nonpositive for any feasible (i.e., satisfying the CTSCBF conditions) choice of L_s, X_s and u_s. Recall the condition (33) gives $\hat{B}_s\hat{B}_s^\top \leq 1$. Thus,

$$\begin{aligned}[\cdots] &= (\gamma-2a)L_s^2 + 2((a-\gamma)v+\beta)L_s + \hat{B}_s^2 - 2\beta v\\ &\leq (\gamma-2a)L_s^2 + 2((a-\gamma)v+\beta)L_s + 1 - 2\beta v.\end{aligned} \tag{47}$$

We will require this bound to be nonpositive for all $L_s \in \mathbb{R}$, which gives the three conditions

$$\gamma - 2a \leq 0, \tag{48}$$

$$1 - 2\beta v \leq 0, \tag{49}$$

$$((a-\gamma)v+\beta)^2 + (2a-\gamma)(1-2\beta v) \leq 0. \tag{50}$$

The first condition (48) is already an assumption. Combining (49) and the condition $\beta \in [0,b]$ gives $1/2v \leq \beta \leq b$, and hence a feasible β exists only if $b \geq 1/2v$, giving $v \geq b/2$. Following the similar procedure to Theorem 2, consider the following QP, which has three solution candidates.

$$\begin{aligned}\beta &= \arg\min_{\frac{1}{2v}\leq\beta\leq b} ((a-\gamma)v+\beta)^2 + (2a-\gamma)(1-2\beta v)\\ &= \begin{cases}\frac{1}{2v} & \text{(case A)}\\ av & \text{(case B)}\\ b & \text{(case C)}\end{cases}\end{aligned} \tag{51}$$

The optimal β for the unconstrained problem is $\beta = av$ here.

Case A: A applies if $av < 1/2v$, i.e., $v < 1/\sqrt{2a}$. The inequality (50) reduces to

$$\left((a-\gamma)v + \frac{1}{2v}\right)^2 \leq 0, \tag{52}$$

which has the only solution $\gamma = a + 1/2v^2$. However, substituting $v < 1/\sqrt{2a}$ yields $\gamma > 2a$, which contradicts to the assumption $\gamma \leq 2a$. Case A is therefore void.

Case B: B applies if $1/2v \leq av \leq b$, i.e., $1/\sqrt{2a} \leq v \leq b/a$. In this case, substituting $\beta = av$ to (50) gives

$$\begin{aligned}0 &\geq (2av - \gamma v)^2 + (2a-\gamma)(1-2v^2)\\ &= (2a-\gamma)(1-\gamma v^2),\end{aligned} \tag{53}$$

which leads to (43).

Case C: Lastly, C applies if $av > b$, i.e., $v > b/a$, this yields (42). □

Similar to the discrete-time case, since (42) is a quadratic equation with respect to γ, we can find a closed-form expression for γ as a piecewise smooth function of v. For many pairs of a and b values, the values of v and γ that satisfies the condition of Theorem 3 is plotted in Fig. 4. It can be clearly seen that the γ value tends to drop with smaller a and higher b values, which aligns with the intuition we started with. It is the same with the discrete-time case that γ eventually reaches 0 with sufficiently large v and b.

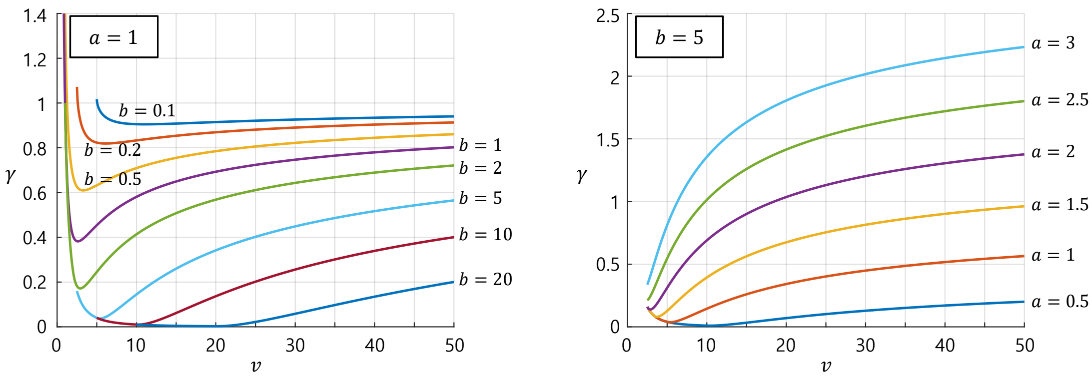

Fig. 4. The plot of γ values as in Theorem 3 (continuous-time version) for many pairs of $\boldsymbol{a}$ and $\boldsymbol{b}$. Smaller γ values denote smaller decay of the survival probability $\boldsymbol{\Gamma}(\boldsymbol{t})$. It can be clearly seen that the behavior follows the intuition that bigger $\boldsymbol{a}$ and $\boldsymbol{b}$ values result in higher chance of survival.

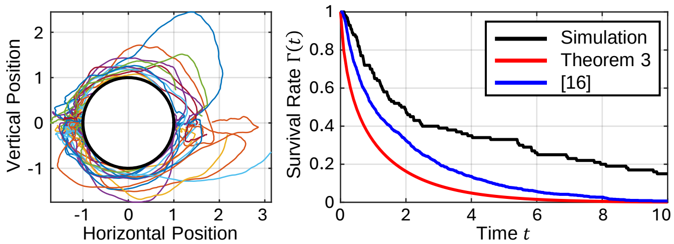

Fig. 5. The simulated trajectories of the continuous-time obstacle avoidance example. It can be seen that due to the unbounded stochasticity of the acceleration, not all trajectories can survive and reach the goal position. Theorem 3 provides a good conservative estimate of the Monte Carlo simulated rate of survival.

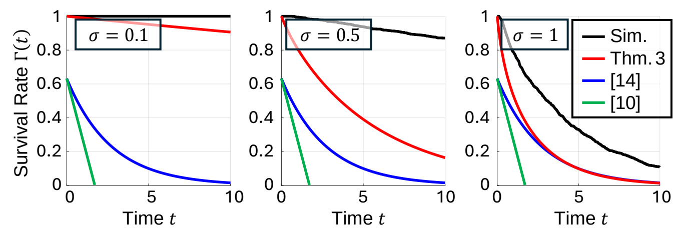

Fig. 6. Comparison with the probability bounds suggested by [14] and [10]. It can be seen that the proposed Theorem 3 provides a significantly tighter bound. The Monte Carlo simulation result (which is identical to [16] in this case) was obtained using 1000 trials for each $\boldsymbol{\sigma}$ value. Sim.=Simulation, Thm.=Theorem.

C. Numerical Example

As a numerical example for the continuous-time case, we consider the following simple double integrator system with acceleration input and disturbance:

$$dX_t = \left(\begin{bmatrix} 0_2 & 1_2 \\ 0_2 & 0_2 \end{bmatrix} X_t + \begin{bmatrix} 0_2 \\ 1_2 \end{bmatrix} u_t \right) dt + \begin{bmatrix} 0_2 \\ \sigma \cdot 1_2 \end{bmatrix} dW_t, \quad (54)$$

where 0_2 and 1_2 denote zero and identity matrices of size 2×2, respectively. The input $u_t \in \mathbb{R}^2$ is assumed to be bounded by $u_t \in [-1,1]\times[-1,1]$ and $\sigma > 0$ is the parameter that describes the amplitude of the two-channel Brownian disturbance W_t. Given the initial condition at $X_0 = [-x_0, 0, 0, 0]^\top$ ($x_0 > 1$), the control objective of this example is to drive the system to the antipodal goal state $X_{\text{goal}} = [x_0, 0, 0, 0]^\top$, while avoiding a unit-disk obstacle whose center is located at the origin. For obstacle avoidance, we build the following CTSCBF candidate:

$$h(x) = \frac{1}{\sigma} \cdot \left(\sqrt{\sqrt{x_1^2 + x_2^2} - 1} + \frac{x_1 x_3 + x_2 x_4}{\sqrt{x_1^2 + x_2^2}} \right), \quad (55)$$

where x_i is the i-th component of x. One can confirm that for any $a > 0$ and $b < 1/2\sigma$, this h is a (a,b)-CTSCBF. The evidence can be found in the appendix. As the reference input signal, we employ a simple proportional-derivative (PD) feedback controller

$$u_{\text{ref}}(X_t, t) = -K(X_t - X_{\text{goal}}) \quad (56)$$

with feedback gain $K = 4 \cdot [1_2, 1_2]$. The CBF-QP-like [18] feedback controller

$$u_t = \underset{u \in U_{\text{ctscbf}}(X_t)}{\arg\min} \ \|u - u_{\text{ref}}(X_t, t)\|^2 \quad (57)$$

was employed. The parameter values $x_0 = 1.1$, $a = 1$, $b = 0.6$, $\sigma = 0.5$ were used.

The Monte Carlo simulation results compared with the probability bound given by Theorem 3 is shown in Fig. 5. In the simulation, we regard the state unsafe if there is no feasible input that satisfies the SCBF constraint, or the trajectory intersects with the obstacle, i.e., $\|X_t\| < 1$. In the plots, we compare with [16] which is the tightest possible bound given the CTSCBF constraints only. Note that this cannot be a fair comparison since the bounds by [16] does not have a closed-form expression and requires Monte Carlo simulations of an Itô SDE.

D. Comparison with Prior Work

Similarly to the discrete-time case, we compare Theorem 3 with [14] and [10] using the following simple one-dimensional stochastic system:

$$dX_t = (-X_t + 1)dt + \sigma dW_t \quad (58)$$

where $\sigma > 0$ is a parameter that indicates disturbance magnitude. A $(1, 1/\sigma)$-CTSCBF $h(x) = x/\sigma$ can be easily found for this system. Since [14] and [10] require a barrier function B (with $B(x) \le 1$ being the safety condition) that is nonnegative everywhere, we used $B(x) = e^{-x}$ for them as suggested by [10, Example 1]. For three different $\sigma \in \{0.1, 0.5, 1\}$ values and starting from the initial condition $X_0 = 1$, the escape probability bounds are depicted in Fig. 6. In the plots, it can be clearly seen that Theorem 3 provides a tighter bound.

IV. Conclusion and Future Work

This paper suggested a lower bound of the probability at which a stochastic system will exit the given set equipped with a valid SCBF for both discrete-time and continuous-time cases. The probability is based on Ville's inequality and a squaring trick was employed to make the bound explicitly depend on the variance (discrete-time) and diffusion rate (continuous-time) of the dynamics. Multiple simulation experiments were conducted to verify the validity of the probability bound, and it was empirically found that the proposed bound is tighter than existing martingale-based bounds.

We suggest three possible future research directions as follows. Firstly, although the probability bounds presented in this paper appear to be tighter than those by existing works, significant conservativeness remains compared to the simulated result. While tighter bounds were obtained by employing the squaring technique, it is an interesting future research direction to consider a wider variety of supermartingale constructions (as in (15) and (44)) which might result in a better bound. Additionally, since the probability bound was derived using the same technique for both discrete- and continuous-time systems, we expect that this paper's result can be extended to stochastic hybrid systems which exhibit both discrete and continuous properties.

Appendix I
The Function h in (28) is a DTSCBF

Given $X_t = [x_1, x_2]^\top$, let $u_t = 1$ and assume $h(X_t) \geq 0$, i.e., $x_1 \leq 25$ and $x_2 \leq 40$. Since h is a Lipschitz continuous function with Lipschitz constant 4, it is easy to find that $\mathrm{Var}[h(X_{t+1})] \leq 4^2 \cdot \mathrm{Var}[X_{t+1}] = 4 \cdot (0.15^2 + 0.2^2) = 1$. Suppose $h(X_t) \geq 0$, and write $X_{t+1} = [x_1', x_2']^\top$. Conditioned on X_t, for the first case $h(X_{t+1}) = 100 - 4x_1'$,

$$\begin{aligned} h(X_{t+1}) &= 100 - 4x_1' \\ &= 100 - 4 \cdot (0.9x_1 + 0.05x_2 + 0.25 + 0.15w_t) \\ &\geq 0.9 \cdot (100 - 4x_1) + 1 - 0.6w_t \\ &\geq 0.9h(X_t) + 1 - 0.6w_t, \end{aligned} \tag{59}$$

and for the second case $h(X_{t+1}) = 160 - 4x_2'$,

$$\begin{aligned} h(X_{t+1}) = 160 - 4x_2' &= 160 - 4 \cdot (0.99x_2 + 0.3 + 0.2z_t) \\ &= 0.99 \cdot (160 - 4x_2) + 0.4 - 0.8z_t \\ &\geq 0.99h(X_t) + 0.4 - 0.8z_t. \end{aligned} \tag{60}$$

Since $\mathbb{E}w_t = \mathbb{E}z_t = 0$, combining the above two gives

$$\mathbb{E}h(X_{t+1}) \geq 0.9h(X_t) + 0.4. \tag{61}$$

Thus, h is a $(0.9, 0.4)$-DTSCBF.

Appendix II
The Function h in (55) is a CTSCBF

Let $u = [x_1, x_2]^\top / \sqrt{x_1^2 + x_2^2} \in [-1, 1] \times [-1, 1]$, and differentiate to get

$$\mathcal{A}h(x, u) = \frac{1}{\sigma}\left(\frac{(B)}{2R \cdot (A)} + \frac{x_3^2 + x_4^2}{R} + 1 - \frac{(B)^2}{R^3}\right), \tag{62}$$

where $(A) = \sqrt{R - 1}$, $(B) = x_1x_3 + x_2x_4$, $R = \sqrt{x_1^2 + x_2^2}$. For any x such that $h(x) = (A) + (B)/R \geq 0$,

$$\mathcal{A}h(x, u) \geq \frac{1}{2\sigma} + \frac{(x_1^2+x_2^2)(x_3^2+x_4^2) - (x_1x_3 + x_2x_4)^2}{\sigma(x_1^2+x_2^2)^{3/2}} \geq \frac{1}{2\sigma}, \tag{63}$$

where the Cauchy-Schwartz inequality was used to obtain the second inequality.

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